Preconditioning of the Two Fluid Equations to Avoid Excessive Dissipation

L'CADAME Weekly Meetings

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Weekly Meetings

Simple Model Problem

Governing Equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0$$
$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2 + p}{\partial x} = 0 \tag{1}$$

Nondimensionalization

Lets scale everything using

$$\overline{\rho} = \frac{\rho}{\rho^*} \quad ; \quad \overline{u} = \frac{u}{u^*}$$
$$\overline{p} = \frac{p}{\rho^* a^{*2}} \quad ; \quad \overline{x} = \frac{x}{\delta^*}$$
$$\overline{t} = \frac{t\delta^*}{u^*} \tag{2}$$

Simple Model Problem

Nondimensionalization

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0$$
$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{1}{M_*^2}\frac{\partial p}{\partial x} = 0 \quad (3)$$

Limit of low Mach number

$$\rho = \rho_0 + \rho_1 M_* + \rho_2 M_*^2 + \ldots$$
$$u = u_0 + u_1 M_* + u_2 M_*^2 + \ldots \quad (4)$$
$$p = p_0 + p_1 M_* + p_2 M_*^2 + \ldots$$

Expanded Equations

M_*^{-2} Terms

$$\frac{\partial p_0}{\partial x} = 0 \tag{5}$$

M_*^{-1} Terms

$$\frac{\partial p_1}{\partial x} = 0 \tag{6}$$

M_*^{0} Terms

$$\frac{\partial \rho_0 u_0}{\partial t} + \frac{\partial \rho_0 u_0^2}{\partial x} + \frac{\partial p_2}{\partial x} = 0 \tag{7}$$

Forcing constant pressure on the boundary will result

$$p(x,t) = p_0(t) + p_2(x,t) M_*^2 + \dots \tag{8}$$

Roe Solver

Governing Equations

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} &= 0 \\ \frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2 + p}{\partial x} &= 0\end{aligned} \tag{9}$$

$$\frac{\partial q}{\partial t} + A\frac{\partial q}{\partial x} = 0 \tag{10}$$

$$q = [\rho, \rho u]^T \tag{11}$$

$$A = \begin{bmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{bmatrix} \tag{12}$$

$$\lambda_{1,2} = u \pm c \quad ; \quad R = \begin{bmatrix} 1 & 1 \\ u - c & u + c \end{bmatrix} \tag{13}$$

Godunov Method

$$\frac{\partial Q_i}{\partial t} + \frac{1}{\Delta x}\left(A^+ \Delta Q_{i-1/2} + A^- \Delta Q_{i+1/2}\right) = 0 \quad (14)$$

$$A^+ \Delta Q_{i-1/2} = \sum_{p=0}^{2} (\lambda^p_{i-1/2})^+ W^p_{i-1/2}$$

$$A^- \Delta Q_{i+1/2} = \sum_{p=0}^{2} (\lambda^p_{i+1/2})^- W^p_{i+1/2} \quad (15)$$

$$\lambda^{\pm} = \frac{1}{2}(\lambda + |\lambda|) \quad ; \quad W^p_{i-1/2} = \beta^p_{i-1/2} r^p_{i-1/2} \quad (16)$$

$$\beta^p_{i-1/2} = R^{-1}_{i-1/2}(Q_i - Q_{i-1}) \quad (17)$$

Godunov Method

$$
\begin{aligned}
\frac{\partial Q_i}{\partial t} + \frac{1}{\Delta x}\left(A^{+}\Delta Q_{i-1/2} + A^{-}\Delta Q_{i+1/2}\right) = 0 \\
A^{+}\Delta Q_{i-1/2} = \frac{1}{4}\rho^{*}u^{*}M\left(\hat{u}_1 + {M^{*}}^{-1}\right) \times \\
\left[(M^{*-1} - \hat{u}_1)\Delta\rho_1 + \Delta(\rho u)_1\right]\begin{bmatrix} 1 \\ u^{*}\hat{u}_1 + c \end{bmatrix} \\
A^{-}\Delta Q_{i+1/2} = \frac{1}{4}\rho^{*}u^{*}M^{*}\left(\hat{u}_2 - M^{*-1}\right) \times \\
\left[(M^{*-1} + \hat{u}_2)\Delta\rho_2 - \Delta(\rho u)_2\right]\begin{bmatrix} 1 \\ u^{*}\hat{u}_2 - c \end{bmatrix}
\end{aligned}
\tag{18}
$$

Final Form of the Equations

$$\frac{\partial \rho}{\partial t} + \frac{\hat{\delta}}{4} M^* \left(\hat{u}_2 - M^{*-1}\right) \left[(M^{*-1} + \hat{u}_2)\Delta\rho_2 - \Delta(\rho u)_2\right] + \frac{\hat{\delta}}{4} M \left(\hat{u}_1 + M^{*-1}\right) \left[(M^{*-1} - \hat{u}_1)\Delta\rho_1 + \Delta(\rho u)_1\right] = 0 \tag{19}$$

$$\frac{\partial \rho u}{\partial t} + \frac{\hat{\delta}}{4} M^* \left(\hat{u}_2 - M^{*-1}\right)^2 \left[(M^{*-1} + \hat{u}_2)\Delta\rho_2 - \Delta(\rho u)_2\right] + \frac{\hat{\delta}}{4} M \left(\hat{u}_1 + M^{*-1}\right)^2 \left[(M^{*-1} - \hat{u}_1)\Delta\rho_1 + \Delta(\rho u)_1\right] = 0 \tag{20}$$

Continuity M_*^{-1}

$$-(\rho_{i+1}^0 - \rho_i^0) + (\rho_i^0 - \rho_{i-1}^0) = 0 \tag{21}$$

Final Form of the Equations

$$\frac{\partial \rho}{\partial t} + \frac{\hat{\delta}}{4} M^* \left(\hat{u}_2 - M^{*-1}\right) \left[(M^{*-1} + \hat{u}_2)\Delta\rho_2 - \Delta(\rho u)_2\right]$$
$$+\frac{\hat{\delta}}{4} M \left(\hat{u}_1 + M^{*-1}\right) \left[(M^{*-1} - \hat{u}_1)\Delta\rho_1 + \Delta(\rho u)_1\right] = 0 \quad (22)$$

$$\frac{\partial \rho u}{\partial t} + \frac{\hat{\delta}}{4} M^* \left(\hat{u}_2 - M^{*-1}\right)^2 \left[(M^{*-1} + \hat{u}_2)\Delta\rho_2 - \Delta(\rho u)_2\right]$$
$$+\frac{\hat{\delta}}{4} M \left(\hat{u}_1 + M^{*-1}\right)^2 \left[(M^{*-1} - \hat{u}_1)\Delta\rho_1 + \Delta(\rho u)_1\right] = 0 \quad (23)$$

Momentum M_*^{-2}

$$(\rho_{i+1}^0 - \rho i^0) + (\rho_i^0 - \rho_{i+1}^0) = 0 \quad (24)$$

Final Form of the Equations

$$
\begin{aligned}
-(\rho^0_{i+1} - \rho^0_i) + (\rho^0_i - \rho^0_{i-1}) = 0 \\
(\rho^0_{i+1} - \rho i'^0) + (\rho^0_i - \rho^0_{i+1}) = 0
\end{aligned}
\tag{25}
$$

Results in

$$
\rho^0_i = \rho^0_{i+1} = \rho^0_{i-1} \tag{26}
$$

$$\frac{\partial \rho}{\partial t} + \frac{\hat{\delta}}{4} M^{*} \left(\hat{u}_2 - M^{*-1}\right) \left[(M^{*-1} + \hat{u}_2)\Delta\rho_2 - \Delta(\rho u)_2\right] + \frac{\hat{\delta}}{4} M \left(\hat{u}_1 + M^{*-1}\right) \left[(M^{*-1} - \hat{u}_1)\Delta\rho_1 + \Delta(\rho u)_1\right] = 0 \tag{27}$$

$$\frac{\partial \rho u}{\partial t} + \frac{\hat{\delta}}{4} M^{*} \left(\hat{u}_2 - M^{*-1}\right)^2 \left[(M^{*-1} + \hat{u}_2)\Delta\rho_2 - \Delta(\rho u)_2\right] + \frac{\hat{\delta}}{4} M \left(\hat{u}_1 + M^{*-1}\right)^2 \left[(M^{*-1} - \hat{u}_1)\Delta\rho_1 + \Delta(\rho u)_1\right] = 0 \tag{28}$$

Momentum M_*^{-1}

$$\rho^1_{i+1} - \rho^1_{i-1} = \rho^0(-u^0_{i+1} + 2u^0_i - u^0_{i-1}) \tag{29}$$

Conclusion

$$
\begin{aligned}
-(\rho_{i+1}^0 - \rho_i^0) + (\rho_i^0 - \rho_{i-1}^0) = 0 \\
(\rho_{i+1}^0 - \rho i^0) + (\rho_i^0 - \rho_{i+1}^0) = 0 \\
\rho_{i+1}^1 - \rho_{i-1}^1 = \rho^0(-u_{i+1}^0 + 2u_i^0 - u_{i-1}^0)
\end{aligned}
\tag{30}
$$

ρ^1 is not zero!

$$
p = p_0 + c^2(\rho - \rho_0) \tag{31}
$$

This means P^1 is not also zero!