Weak Formulation of Roe-7 Scheme

L'CADAME Weekly Meetings

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Weekly Meetings

ROE-5 Equations

Governing equation

- Generic equations for 1-dimensional iso-thermal 2-fluid model

$$\frac{\partial \alpha_g}{\partial t} + u_i \frac{\partial \alpha_g}{\partial x} = 0 \tag{1}$$

$$\frac{\partial \alpha_k \rho_k}{\partial t} + \frac{\partial \alpha_k \rho_k u_k}{\partial x} = 0 \tag{2}$$

$$\frac{\partial \alpha_k \rho_k u_k}{\partial t} + \frac{\partial \alpha_k \rho_k u_k^2 + \alpha_k P_k}{\partial x} - P_{ik} \frac{\partial \alpha_k}{\partial x} = 0 \tag{3}$$

- For heat transfer, we need an energy equation for each phase.

Different forms of Energy equation

- Thermodynamics relations

$$E_k = e_k + \frac{1}{2}u_k^2 \quad ; \quad h_k = e_k + \frac{P_k}{\rho_k}$$

- Total energy

$$\frac{\partial \alpha_k \rho_k E_k}{\partial t} + \frac{\partial \alpha_k \rho_k E_k u_k + \alpha_k P_k u_k}{\partial x} + P_{ik}\frac{\partial \alpha_k}{\partial t} = 0 \tag{4}$$

- Internal energy

$$\frac{\partial \alpha_k \rho_k e_k}{\partial t} + P_{ik}\frac{\partial \alpha_k}{\partial t} + \frac{\partial \alpha_k \rho_k e_k u_k}{\partial x} + P_{ik} u_k \frac{\partial \alpha_k}{\partial x} + \frac{P_k}{\rho_k}\left(\frac{\partial \alpha_k \rho_k u_k}{\partial x} - u_k \frac{\partial \alpha_k \rho_k}{\partial x}\right) = 0 \tag{5}$$

- Enthalpy

$$\frac{\partial \alpha_k \rho_k h_k}{\partial t} - \Delta P_k \frac{\partial \alpha_k}{\partial t} - \alpha_k \frac{\partial P_k}{\partial t} + \frac{\partial \alpha_k \rho_k h_k u_k}{\partial x} - u_k \Delta P_k \frac{\partial \alpha_k}{\partial x} - \alpha_k u_k \frac{\partial P_k}{\partial x} = 0 \tag{6}$$

$\Rightarrow \Delta P_k = P_k - P_{ik}.$

ROE-7 Hyperbolicity

Roe-7 model well-posedness

- To have a well-posed model, the system of equations (SoE) must be hyperbolic. E.g for SoE using total energy, we have

$$\lambda = [u_g - c_g,\ u_g + c_g,\ u_g,\ u_i,\ u_l - c_l,\ u_l + c_l,\ u_l]$$

$$\begin{bmatrix} 1 & 1 & 1 & r_{14} & 0 & 0 & 0 \\ u_g - c_g & u_g + c_g & u_g & r_{14} u_i & 0 & 0 & 0 \\ H_g - u_g c_g & H_g + u_g c_g & H_g - \frac{c_g^2}{\Gamma_g} & r_{34} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_{54} & 1 & 1 & 0 \\ 0 & 0 & 0 & r_{54} u_i & u_l - c_l & u_l + c_l & u_l \\ 0 & 0 & 0 & r_{74} & H_l - u_l c_l & H_l + u_l c_l & H_l - \frac{c_l^2}{\Gamma_l} \end{bmatrix}$$

$$r_{14} = \frac{\rho_g c_g^2 - (\Gamma_g + 1)(P_g - P_{ig})}{c_g^2 - (u_g - u_i)^2}, \quad r_{34} = (H_g - u_g^2 + u_g u_i)\, r_{14} - P_{ig}$$

$$r_{54} = -\frac{\rho_l c_l^2 - (\Gamma_l + 1)(P_l - P_{il})}{c_l^2 - (u_l - u_i)^2}, \quad r_{74} = -(H_l - u_l^2 + u_l u_i)\, r_{54} + P_{il}$$

ROE Linearization

Weak Formulation of Roe Scheme

- Since, the governing SoE, $\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x} = \mathbf{0}$ is non-conservative, the ROE method cannot be directly used.

$$\mathbf{U} = \begin{bmatrix} \alpha_g \rho_g & \alpha_g \rho_g u_g & \alpha_g \rho_g E_g & \alpha_g & \alpha_l \rho_l & \alpha_l \rho_l u_l & \alpha_l \rho_l E_l \end{bmatrix}^T$$

- Solution: The weak formulation of Roe $\Rightarrow$ conservative SoE $\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \mathbf{0}$ is approximated by a linear equations system:

$$\frac{\partial \mathbf{U}}{\partial t} + \tilde{\mathbf{A}}(\mathbf{U}_L, \mathbf{U}_R)\frac{\partial \mathbf{U}}{\partial x} = \mathbf{0} \tag{7}$$

$\Rightarrow$ $\tilde{\mathbf{A}}$ is the averaged matrix, subscripts L and R denote left and right.

- Averaged matrix must satisfy 3 conditions:
  1. Hyperbolicity condition. All eigenvalues of $\tilde{\mathbf{A}}$ are real.
  2. Consistency condition: $\tilde{\mathbf{A}}(\mathbf{U}_L, \mathbf{U}_R) = \mathbf{A}(\mathbf{U})$
  3. Jump condition: $\mathbf{F}(\mathbf{U}_R) - \mathbf{F}(\mathbf{U}_L) = \tilde{\mathbf{A}}(\mathbf{U}_L, \mathbf{U}_R)(\mathbf{U}_R - \mathbf{U}_L)$

ROE Linearization

ROE-7 Averaged Matrix

- Based on Toumi 92, for a canonical integral path of

$$\mathbf{\Phi}(s; \mathbf{U}_L, \mathbf{U}_R) = \mathbf{U}_L + s(\mathbf{U}_L - \mathbf{U}_R) \tag{8}$$

the averaged matrix is obtained such that

$$\tilde{\mathbf{A}}(\mathbf{U}_L, \mathbf{U}_R) = \int_0^1 \mathbf{A}(\mathbf{\Phi}(s; \mathbf{U}_L, \mathbf{U}_R))\, ds \tag{9}$$

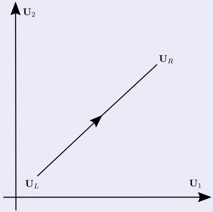

Fig. 1: Integral path

- Once the eigenvalues and right eigenvectors of $\tilde{\mathbf{A}}$ are available, the Riemann problem with the approximated linear equations system can be solved by Godunov method:

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}\left(\tilde{\mathbf{A}}^-_{i+1/2}(\mathbf{U}^n_{i+1} - \mathbf{U}^n_i) + \tilde{\mathbf{A}}^+_{i-1/2}(\mathbf{U}^n_i - \mathbf{U}^n_{i-1})\right) \tag{10}$$

ROE Linearization

ROE-7 Averaged Matrix

- For our SoE, the averaged matrix $\tilde{\mathbf{A}}$ is

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} & 0 & 0 & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{u}_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & A_{64} & A_{65} & A_{66} & A_{67} \\ 0 & 0 & 0 & A_{74} & A_{75} & A_{76} & A_{77} \end{bmatrix}$$

$$\begin{aligned} A_{21} &= \frac{\hat{\gamma}_g - 3}{2}\hat{u}_g^2 + (\hat{\gamma}_g - 1)\beta_g & A_{22} &= (3 - \hat{\gamma}_g)\hat{u}_g & A_{23} &= \hat{\gamma}_g - 1 \\ A_{65} &= \frac{\hat{\gamma}_l - 3}{2}\hat{u}_l^2 + (\hat{\gamma}_l - 1)\beta_l & A_{66} &= (3 - \hat{\gamma}_l)\hat{u}_l & A_{67} &= \hat{\gamma}_l - 1 \\ A_{31} &= (\frac{\hat{\gamma}_g - 1}{2}\hat{u}_g^2 - \hat{H}_g + (\hat{\gamma}_g - 1)\beta_g)\hat{u}_g & A_{32} &= \hat{H}_g - (\hat{\gamma}_g - 1)\hat{u}_g^2 & A_{33} &= \hat{\gamma}_g\hat{u}_g \\ A_{75} &= (\frac{\hat{\gamma}_l - 1}{2}\hat{u}_l^2 - \hat{H}_l + (\hat{\gamma}_l - 1)\beta_l)\hat{u}_l & A_{76} &= \hat{H}_l - (\hat{\gamma}_l - 1)\hat{u}_l^2 & A_{77} &= \hat{\gamma}_l\hat{u}_l, \end{aligned}$$

$$\begin{aligned} A_{24} &= (\hat{\gamma}_g - 1)\bar{P}_g - \hat{\rho}_g\hat{C}_g^2 + \bar{P}_g - \bar{P}_{ig} & A_{34} &= (\hat{\gamma}_g\bar{p}_g - \hat{\rho}_g\hat{C}_g^2)\hat{u}_g - \overline{u_i P_{ig}} \\ A_{64} &= \hat{\rho}_l\hat{C}_l^2 - (\hat{\gamma}_l - 1)\bar{P}_l - (\bar{P}_l - \bar{P}_{il}) & A_{74} &= (\hat{\rho}_l\hat{C}_l^2 - \hat{\gamma}_l\bar{P}_l)\hat{u}_l + \overline{u_i P_{il}} \end{aligned}$$

which gives 7 distinct eigenvalues:

$$\lambda = [\hat{u}_g - \hat{C}_g,\ \hat{u}_g + \hat{C}_g,\ \hat{u}_g,\ \bar{u}_i,\ \hat{u}_l - \hat{C}_l,\ \hat{u}_l + \hat{C}_l,\ \hat{u}_l]$$

3-Fluid Model

Governing Equation

$$\frac{\partial \alpha_1 \rho_1}{\partial t} + \frac{\partial \alpha_1 \rho_1 u_1}{\partial x} = 0 \qquad (11)$$

$$\frac{\partial \alpha_1 \rho_1 u_1}{\partial t} + \frac{\partial \alpha_1 \rho_1 u_1^2 + \alpha_1 P_1}{\partial x} - P_{i12} \frac{\partial \alpha_1}{\partial x} = 0 \qquad (12)$$

$$\frac{\partial \alpha_1 \rho_1 E_1}{\partial t} + \frac{\partial \alpha_1 \rho_1 E_1 u_1 + \alpha_1 P_1 u_1}{\partial x} + P_{i12} \frac{\partial \alpha_1}{\partial t} = 0 \qquad (13)$$

$$\frac{\partial \alpha_3 \rho_3}{\partial t} + \frac{\partial \alpha_3 \rho_3 u_3}{\partial x} = 0 \qquad (14)$$

$$\frac{\partial \alpha_3 \rho_3 u_3}{\partial t} + \frac{\partial \alpha_3 \rho_3 u_3^2 + \alpha_3 P_3}{\partial x} - P_{i32} \frac{\partial \alpha_3}{\partial x} = 0 \qquad (15)$$

$$\frac{\partial \alpha_3 \rho_3 E_3}{\partial t} + \frac{\partial \alpha_3 \rho_3 E_3 u_3 + \alpha_3 P_3 u_3}{\partial x} + P_{i32} \frac{\partial \alpha_3}{\partial t} = 0 \qquad (16)$$

$$\frac{\partial \alpha_2 \rho_2}{\partial t} + \frac{\partial \alpha_2 \rho_2 u_2}{\partial x} = 0 \qquad (17)$$

$$\frac{\partial \alpha_2 \rho_2 u_2}{\partial t} + \frac{\partial \alpha_2 \rho_2 u_2^2 + \alpha_2 P_2}{\partial x} - (P_{i21} + P_{i23}) \frac{\partial \alpha_2}{\partial x} = 0 \qquad (18)$$

$$\frac{\partial \alpha_2 \rho_2 E_2}{\partial t} + \frac{\partial \alpha_2 \rho_2 E_2 u_2 + \alpha_2 P_2 u2}{\partial x} + (P_{i21} + P_{i23}) \frac{\partial \alpha_2}{\partial t} = 0 \qquad (19)$$

$$\frac{\partial \alpha_1}{\partial t} + u_{i12} \frac{\partial \alpha_1}{\partial x} = 0 \qquad (20)$$

$$\frac{\partial \alpha_2}{\partial t} + u_{i23} \frac{\partial \alpha_2}{\partial x} = 0 \qquad (21)$$

$\Rightarrow$ complemented by $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

Hyperbolicity

- 11 distinct eigenvalues and eigenvectors ⇒ SoE is hyperbolic.

$$\lambda = [u_1 - c_1,\ u_1 + c_1,\ u_1,\ u_{i12},\ u_2 - c_2,\ u_2 + c_2,\ u_2,\ u_{i23},\ u_3 - c_3,\ u_3 + c_3,\ u_3]$$

$$\mathbb{R} = \begin{bmatrix} 1 & 1 & 1 & r_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_1 - c_1 & u_1 + c_1 & u_1 & u_{i12}\ r_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ r_{31} & r_{32} & r_{33} & r_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & r_{58} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_2 - c_2 & u_2 + c_2 & u_2 & u_{i23}\ r_{58} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_{75} & r_{76} & r_{77} & r_{78} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_{94} & 0 & 0 & 0 & r_{98} & 1 & 1 & 1 \\ 0 & 0 & 0 & u_{i12}\ r_{94} & 0 & 0 & 0 & u_{i23}\ r_{98} & u_3 - c_3 & u_3 + c_3 & u_3 \\ 0 & 0 & 0 & r_{114} & 0 & 0 & 0 & r_{118} & r_{119} & r_{1110} & r_{1111} \end{bmatrix}$$

$$\begin{array}{lll} r_{31} = H_1 - u_1 c_1 & r_{75} = H_2 - u_2 c_2 & r_{119} = H_3 - u_3 c_3 \\ r_{32} = H_1 + u_1 c_1 & r_{76} = H_2 + u_2 c_2 & r_{1110} = H_3 + u_3 c_3 \\ r_{33} = H_1 - c_1^2/\Gamma_1 & r_{77} = H_2 - c_2^2/\Gamma_2 & r_{1111} = H_3 - c_3^2/\Gamma_3 \end{array}$$

$$r_{14} = \rho_1 c_1^2 - (\Gamma_1 + 1)(P_1 - P_{i12})/(c_1^2 - (u_1 - u_{i12})^2), \quad r_{34} = (H_1 - u_1^2 - u_1\ u_{i12})\ r_{14} - P_{i12}$$

$$r_{94} = -(\rho_3 c_3^2 - (\Gamma_3 + 1)(P_3 - P_{i32}))/(c_3^2 - (u_3 - u_{i12})^2), \quad r_{114} = -(H_3 - u_3^2 - u_3\ u_{i12})\ r_{94} + P_{i32}$$

$$r_{58} = \rho_2 c_2^2 - (\Gamma_2 + 1)(P_3 - P_{i21} - P_{i23})/(c_2^2 - (u_2 - u_{i23})^2), \quad r_{78} = (H_2 - u_2^2 - u_2\ u_{i23})\ r_{14} - P_{i21} - P_{i23}$$

$$r_{98} = -(\rho_3 c_3^2 - (\Gamma_3 + 1)(P_3 - P_{i32}))/(c_3^2 - (u_3 - u_{i23})^2), \quad r_{118} = -(H_3 - u_3^2 - u_3\ u_{i23})\ r_{98} + P_{i32}$$